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One-one, onto functions and bijections

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When is a function said to be one-one? If different elements of the domain have different images. What does this mean? If we take two elements x_1 and x_2 in the domain, and $x_1 \neq x_2$, then we must have the images of x_1 and x_2 are different. Let us write this formally:

Definition: A function $f : X \to Y$ is said to be *one-one* (or one-to-one or injective) if for each pair of points $x_1, x_2 \in X$ with $x_1 \neq x_2$, we have $f(x_1) \neq f(x_2)$.

Example 1. The identity function on any nonempty set is one-one.

Example 2. The function $f : (0, 1) \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ is one-one.

Proof: For any $x_1, x_2 \in (0, 1)$, if $x_1 \neq x_2$, then we have $\frac{1}{x_1} \neq \frac{1}{x_2}$, that is, $f(x_1) \neq f(x_2)$. Therefore, f is one-one.

Example 3. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ is one-one.

Proof: Suppose $x, y \in \mathbb{R}$ with $x \neq y$. We claim that $f(x) \neq f(y)$, i.e., $x^3 \neq y^3$. This is equivalent to showing that $x^3 - y^3 \neq 0$. Note that, $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$. We consider different cases:

(1) If x and y are both positive or both negative, then $x^2 + xy + y^2 > 0$. Since $x - y \neq 0$, this implies $x^3 - y^3 \neq 0$.

(2) If one of x and y is zero, then the other is nonzero. Thus, one of x^3 and y^3 is zero and the other is nonzero, hence they are unequal.

(3) If x and y are of opposite signs, so are x^3 and y^3, and hence $x^3 \neq y^3$.

Thus our claim holds, and therefore f is one-one.

Proposition 1 A function $f : X \to Y$ is one-one if and only if

$$\forall x_1, x_2 \in X,\ f(x_1) = f(x_2) \implies x_1 = x_2.$$

Proposition 1 is very useful in proving that a function is one-one.

Example 1 The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 3x + 2$ is one-one.

Proof: For $x_1, x_2 \in \mathbb{R}$, we have

$$f(x_1) = f(x_2) \quad \implies \quad 3x_1 + 2 = 3x_2 + 2 \quad \implies \quad x_1 = x_2.$$

Therefore, f is one-one.

Example 2: The function $f : [0, \pi) \to \mathbb{R}$ given by $f(x) = \cos x$ is one-one.

Proof: Suppose $x, y \in [0, \pi)$ and $f(x) = f(y)$, i.e., $\cos x = \cos y$. Without loss of generality, assume $x \geq y$. Now, we have $0 = \cos x - \cos y = -2\sin\left(\frac{x-y}{2}\right)\sin\left(\frac{x+y}{2}\right)$. This implies $\sin\left(\frac{x-y}{2}\right) = 0$ or $\sin\left(\frac{x+y}{2}\right) = 0$. Since $x, y \in [0, \pi)$ and $x \geq y$, we have $0 \leq \frac{x-y}{2}, \frac{x+y}{2} < \pi$. Because $\sin\theta > 0$ for $0 < \theta < \pi$, the only possibilities are $\frac{x-y}{2} = 0$ or $\frac{x+y}{2} = 0$, that is, $x - y = 0$ or $x + y = 0$. Since $x, y \geq 0$, the condition $x + y = 0$ implies $x = y = 0$. Hence, in all cases, $f(x) = f(y)$ implies $x = y$. Therefore, f is one-one.

Exercise

Exercise 1 Show that the following functions are one-one.

(i) $f : [0, 1) \to \mathbb{R}$ defined by $f(x) = x^2$.

(ii) $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(k) = 3k + 7$.

(iii) $f : \mathbb{N} \to \mathbb{N}$ defined by

$$f(n) = \begin{cases} n + 2, & \text{if } n \text{ is odd,} \\ 2n, & \text{if } n \text{ is even.} \end{cases}$$

Hints:

(i) Using the definition: assume $x_1, x_2 \in [0, 1)$ and $x_1 \neq x_2$. Show that $x_1^2 \neq x_2^2$ (use the fact that in $[0, 1)$, squaring is strictly increasing).

(ii) Using the definition: assume $k_1, k_2 \in \mathbb{Z}$ and $k_1 \neq k_2$. Show that $3k_1 + 7 \neq 3k_2 + 7$.

(iii) Consider different cases based on parity (odd or even) of $n_1, n_2 \in \mathbb{N}$. Show that in all cases, $f(n_1) = f(n_2)$ implies $n_1 = n_2$.

When is a function not one-one?

A function $f : X \to Y$ is *not one-one* if we can find elements $x_1, x_2 \in X$ such that $x_1 \neq x_2$ and $f(x_1) = f(x_2)$.

Example 1 The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is not one-one.

Proof: Take $1, -1 \in \mathbb{R}$. We have

$$f(-1) = (-1)^2 = 1.$$
$$f(1) = 1^2 = 1.$$

Thus, $f(-1) = f(1)$ even though $-1 \neq 1$. Therefore, f is not one-one.

Exercise 1. Let $M(2,\mathbb{R})$ denote the set of all 2×2 matrices over $\mathbb{R}$. Consider the function $f : M(2,\mathbb{R}) \to \mathbb{R}$ defined by $f(A) = \det(A)$. Show that f is not one-one.

Solution. Take two matrices $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. in $M(2,\mathbb{R})$. We have

$$\det(A) = 1 \cdot 1 - 0 \cdot 0 = 1, \qquad \det(B) = 1 \cdot 1 - 1 \cdot 0 = 1.$$

Thus $\det(A) = \det(B)$ although $A \neq B$. Therefore f is not one-one.

Onto function

A function $f : X \to Y$ is said to be onto if the codomain of f equals its range, that is, if every element in Y is the image of some element in X, that is, if every element $y \in Y$ has a preimage. Its formal definition is given below:

Definition (Onto function): A function $f : X \to Y$ is said to be *onto* or *surjective* if for each $y \in Y$, there exists $x \in X$ such that $y = f(x)$. In symbols,

$$f : X \to Y \text{ is onto if } \forall y \in Y,\ \exists x \in X \text{ such that } y = f(x).$$

Example 1: The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x + 2$ is onto.

Proof: Let $y \in \mathbb{R}$. We need to find $x \in \mathbb{R}$ such that $f(x) = 3x + 2 = y$. Solving for x, we get $x = \frac{y-2}{3}$. Clearly, $x \in \mathbb{R}$, and $f(x) = y$. Hence, f is onto.

When is a function $f : X \to Y$ *not onto*?

A function $f : X \to Y$ is not onto if $\exists y \in Y$ such that $\forall x \in X,\ y \neq f(x)$.

In other words, f is not onto if there is an element $y \in Y$ which does not have a preimage in X, that is, no element of X maps to y. Alternately, f is not onto if the range $R(f)$ is a proper subset of Y, i.e., $R(f) \subsetneq Y$.

Example 1 The function $f : \mathbb{Z} \to \mathbb{N}$ defined by $f(m) = m^2$ is not onto.

Proof: We claim that $2 \in \mathbb{N}$ does not have a preimage. To see this, we need to show that for any integer m, $f(m) = m^2 \neq 2$.

(1) If $m \in (-\infty, -2] \cup [2, \infty)$, then $f(m) = m^2 \geq 4 > 2$.

(2) For $m \in \{-1, 0, 1\}$, $f(m) \in \{0, 1\} \neq 2$.

Thus, $f(m) \neq 2$ for each integer m, and hence 2 has no preimage. Therefore, f is not onto.

Example 2 The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + x + 1$ is not onto.

Proof: We claim that $0 \in \mathbb{R}$ does not have a preimage. On the contrary, assume that x is the preimage of 0. Then $f(x) = 0$, which implies that $f(x) = x^2 + x + 1 = x^2 + 2x + 1 - x = (x+1)^2 - x = 0 \implies (x+1)^2 = x.$

This implies $x \geq 0$. But, if $x \geq 0$, then

$$f(x) = x^2 + x + 1 \geq 1,$$

which is a contradiction.

Hence, 0 does not have a preimage and therefore f is not onto.

Exercise 1. Determine which of the following functions are surjective:

(a) $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = e^x$.

(b) $f : (0, 1) \to \mathbb{R}$ where $f(x) = \log x$.

(c) $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = \sin x$.

Hints:

- (a) The exponential function $e^x > 0$ for all $x \in \mathbb{R}$, so it does not produce negative numbers. Hence, f is *not onto* $\mathbb{R}$.
- (b) The natural logarithm $\log x$ is defined for $x \in (0, 1)$ and its range is $(-\infty, 0)$. Since $(-\infty, 0) \subsetneq \mathbb{R}$, f is *not onto* $\mathbb{R}$.
- (c) The sine function satisfies $-1 \leq \sin x \leq 1$ for all $x \in \mathbb{R}$. Since the codomain is $\mathbb{R}$, f is *not onto* $\mathbb{R}$.

Definition (Bijection). A function $f : X \to Y$ which is both one-one and onto is called a *bijection* or a *one-one onto function*.

Example 1. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x+2$ is a bijection.

Proof: Step 1: f is one-one.

For $x_1, x_2 \in \mathbb{R}$, we have

$$f(x_1) = f(x_2) \implies 3x_1 + 2 = 3x_2 + 2 \implies x_1 = x_2.$$

Therefore, f is one-one.

Step 2: f is onto.

Let $y \in \mathbb{R}$. We need to find $x \in \mathbb{R}$ such that $f(x) = 3x + 2 = y$. Solving for x, we get $x = \frac{y-2}{3}$. Clearly, $x \in \mathbb{R}$, and $f(x) = y$. Hence, f is onto.

Since f is both one-one and onto, f is a bijection.

Example Consider the following sets of natural numbers:

$$E = \{n \in \mathbb{N} : n = 2k \text{ for some } k \in \mathbb{N}\}, \quad \text{the set of even natural numbers,}$$

$$O = \{n \in \mathbb{N} : n = 2k - 1 \text{ for some } k \in \mathbb{N}\}, \quad \text{the set of odd natural numbers.}$$

We investigate whether there is a bijection between the following pairs:

(1) $\mathbb{N}$ and O

(2) $\mathbb{N}$ and E

(3) E and O

Step 1: Bijection between $\mathbb{N}$ and O

Define the function $f : \mathbb{N} \to O$ by

$$f(n) = 2n - 1.$$

- *Injective:* If $f(n_1) = f(n_2)$, then $2n_1 - 1 = 2n_2 - 1 \implies n_1 = n_2$.
- *Surjective:* For any $m \in O$, $m = 2k - 1$ for some $k \in \mathbb{N}$. Then $f(k) = m$.

Hence, f is a bijection.

Example Continued

Step 2: Bijection between $\mathbb{N}$ and E

Define the function $g : \mathbb{N} \to E$ by $g(n) = 2n$.

- *Injective:* If $g(n_1) = g(n_2)$, then $2n_1 = 2n_2 \implies n_1 = n_2$.
- *Surjective:* For any $m \in E$, $m = 2k$ for some $k \in \mathbb{N}$. Then $g(k) = m$.

Hence, g is a bijection.

Step 3: Bijection between E and O

Define the function $h : E \to O$ by $h(n) = n - 1$.

- *Injective:* If $h(n_1) = h(n_2)$, then $n_1 - 1 = n_2 - 1 \implies n_1 = n_2$.
- *Surjective:* For any $m \in O$, $m = 2k - 1$ for some $k \in \mathbb{N}$. Then $n = m + 1 = 2k \in E$, and $h(n) = m$.

Hence, h is a bijection.

Exercise

Exercise 1 Exhibit a bijection between the following sets:

(1) $\mathbb{N}$ and $\{0\} \cup \mathbb{N}$.

(2) $\mathbb{N}$ and $\mathbb{Z}$.

(3) $\mathbb{N}$ and $\{m \in \mathbb{Z} : m \geq m_0\}$, where m_0 is a fixed integer.

(4) $E = \{2, 4, 6, 8, \ldots\}$ and $\{0\} \cup \mathbb{N}$.

Hints:

(1) Consider $f : \mathbb{N} \to \{0\} \cup \mathbb{N}$ defined by $f(n) = n - 1$.

(2) Consider $f : \mathbb{N} \to \mathbb{Z}$ defined by $f(n) = \begin{cases} -\frac{(n+1)}{2}, & \text{if } n \text{ is odd,} \\ \frac{n}{2} - 1, & \text{if } n \text{ is even.} \end{cases}$

(3) Consider $f : \mathbb{N} \to \{m \in \mathbb{Z} : m \geq m_0\}$ defined by $f(n) = m_0 + n - 1$.

(4) Consider $f : \mathbb{E} \to \{0\} \cup \mathbb{N}$ defined by $f(n) = \frac{n}{2} - 1$.

Exercise 2: Let O be the set of all odd natural numbers and E be the set of all even natural numbers. Give an example of a function $f : O \to E$ which is:

(a) One-one but not onto.

(b) Onto but not one-one.

(c) Neither one-one nor onto.

(d) One-one and onto (bijection).

Exercise 3: Find a one-one map from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$ and a one-one map from $\mathbb{N}$ to $\mathbb{N} \times \mathbb{N}$.

Hint: Consider the function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $f(m, n) = 2^m 3^n$ and Consider another function $g : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ defined by $f(n) = (n, n)$.

Exercise 4: Give a one-one map from the set of rational numbers $\mathbb{Q}$ to $\mathbb{Z} \times \mathbb{N}$.

Hint:

- Any rational number can be written as p/q in lowest terms, where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$.
- Define $f : \mathbb{Q} \to \mathbb{Z} \times \mathbb{N}$ by $f\left(\frac{p}{q}\right) = (p, q)$, which is injective because the reduced fraction representation is unique.

Exercise 5: Determine whether the following functions are one-one and/or onto.

(a) $f : \mathbb{N} \to \mathbb{N}$,

$$f(n) = \begin{cases} n, & \text{if } n \text{ is odd,} \\ 2n, & \text{if } n \text{ is even.} \end{cases}$$

(b) $f : [0, 1] \to [0, 1], f(x) = \frac{1-x}{1+x}$.

(c) $f : [0, 1] \to [a, b], f(x) = bx + a(1 - x)$.

(d) $f : \mathbb{R} \to \mathbb{R}, f(x) = \frac{1}{2}(x + |x|)$.

(e) $f : \mathbb{R} \to \mathbb{R}, f(x) = x + [x]$, where $[x]$ denotes the greatest integer less than or equal to x.

(f) $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} x^2 + x + 1, & x \geq 0, \\ x + 1, & x < 0. \end{cases}$$

(g) $f : [0, 2\pi) \to D = \{(x, y) : x^2 + y^2 = 1\}, f(x) = (\cos x, \sin x)$.

(h) $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2$.

(i) $g : \mathbb{R} \to \mathbb{R}, g(x) = x^3$.